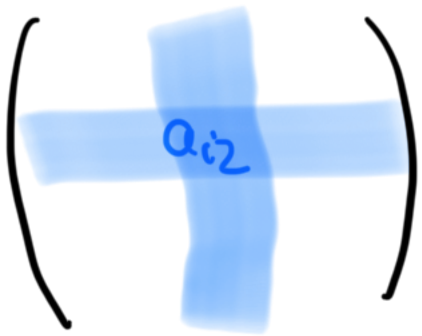
Last time: determinants by (Laplace) cofactor expansion

$$\det(A) = \det\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} =$$

$$a_{i1}\underbrace{(-1)^{i+1}\det(A_{i1})}_{\text{this is called a cofactor}} + a_{i2}\underbrace{(-1)^{i+2}\det(A_{i2})}_{\text{this is called a cofactor}} + \dots + a_{in}\underbrace{(-1)^{i+n}\det(A_{in})}_{\text{this is called a cofactor}}$$

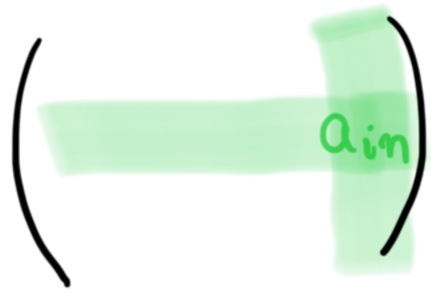
where A_{i1} = A without row i column 1 =

A_{i2} = A without row i column 2 =



$\vdots$

A_{in} = A without row i column n =



(also works for columns instead of rows)

- Recall the main tool that went into proof: the function

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \Phi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ x_1 & x_2 & \cdots & x_n \\ a_{i+1,1} & a_{i+1,2} & & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is linear, i.e.

$$\begin{cases} \Phi\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} = \Phi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \Phi\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\ \Phi\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = \lambda\, \Phi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{cases}$$

- Example (det of triangular matrices by cofactor expansion)

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & 0 & & \ddots & \vdots \\ & & & & a_{nn} \end{pmatrix} \overset{\text{cofactor expansion along } n\text{-th row}}{=} 0\cdot\sqcup + 0\cdot\sqcup + \cdots + 0\cdot\sqcup + a_{nn}\cdot\underbrace{\sqcup}$$

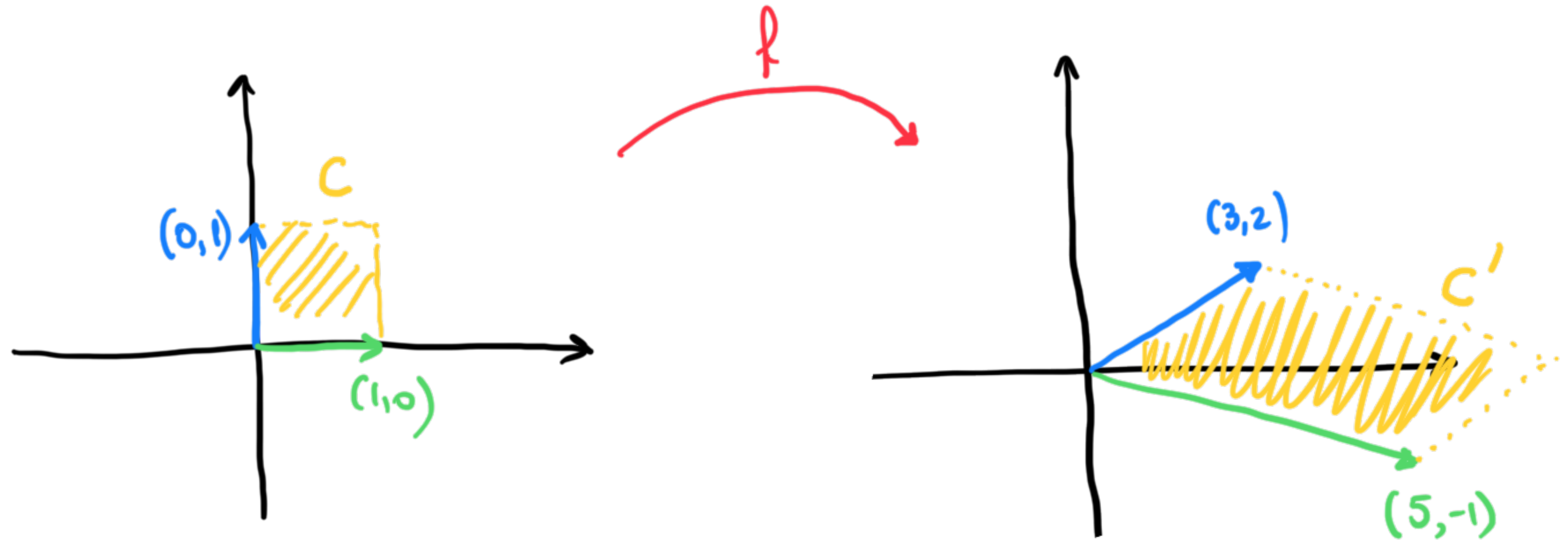
$$(-1)^{n+n} \det\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{pmatrix}$$

$$= a_{nn} \det\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{n-1,n-1} \end{pmatrix} = a_{nn}\, a_{n-1,n-1} \det\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{n-2,n-2} \end{pmatrix}$$

$$\overset{\text{induction}}{\underset{\text{recursion}}{=\!=\!=}} a_{nn}\, a_{n-1,n-1} \cdots a_{22}\, a_{11}$$

How to use determinants to calculate areas / volumes?

①



$\text{area}(C) = 1$ $\qquad\qquad$ $\text{area}(C') = ?$

choose $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$$f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

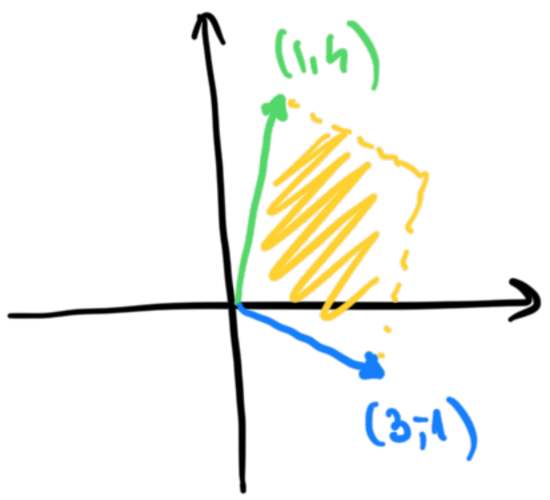
$$f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$13 = 5 \cdot 2 - (-1)3 = \det(A) = \frac{\text{area}(C')}{\text{area}(C)} = \frac{\text{area}(C')}{1}$$

$$\Downarrow$$

$$\text{area}(\text{parallelogram}) = 13$$

②

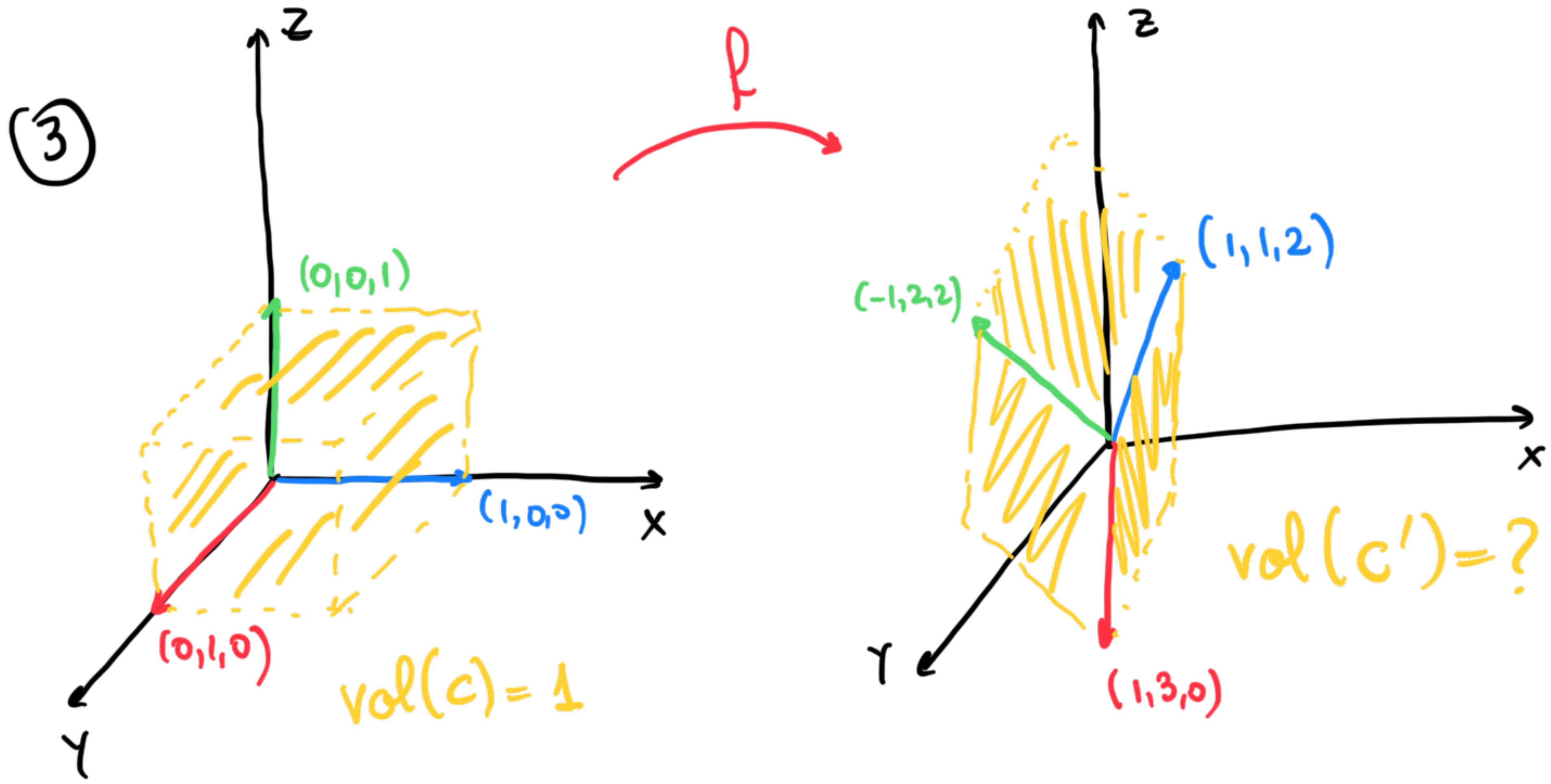


$$A = \begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix}$$

$$\det(A) = 1 \cdot (-1) - 3 \cdot 4 = -13$$

In this case, $\text{area}(\ \text{[shaded parallelogram]}\ ) = |\det(A)| = 13$

③



define $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\2\end{pmatrix}$

$$f\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

$$f\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & 1 & -1\\1 & 3 & 2\\2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\frac{\mathrm{vol}(c')}{\mathrm{vol}(c)=1} = \det(A) = \det\begin{pmatrix}1 & 1 & -1\\1 & 3 & 2\\2 & 0 & 2\end{pmatrix} = 14.$$

Note: if you want to compute actual volumes (not these funny signed volumes) you need to replace $\det(A)$ by $|\det(A)|$ above.

New topic: Vector spaces

elements of V are called "vectors"

DEF 12.1: A **vector space** is a set V together with two operations together with ...

- "addition": $\forall\, v, w \in V,\ \exists\ v+w \in V$
- "scalar multiplication": $\forall v \in V, \forall \lambda \in \mathbb{R},\ \exists\ \lambda v \in V$

satisfying the following axioms:

- $v + w = w + v$ (commutativity)
- $(v+w) + u = v + (w+u)$ (associativity)
- $\exists$ a special element $0_V \in V$ s.t. $0_V + v = v + 0_V = v$ (neutral element)
- $\forall\, v \in V,\ \exists\ -v$ s.t. $v + (-v) = (-v) + v = 0_V$ (opposite vector)
- $(\lambda + \mu)\, v = \lambda v + \mu v$ (distributivity 1)
- $\lambda (v + w) = \lambda v + \lambda w$ (distributivity 2)
- $(\lambda \mu)\, v = \lambda (\mu v)$
- $1 v = v$

Ex: $V = \mathbb{R}^n$ with usual addition and scalar multiplication

Ex: $\mathbb{R}^{m \times n}$ ———— // ————

$A + B$ = component-wise addition

λA = usual multiplication

Ex: $W = \{ \text{linear } f : \mathbb{R}^n \to \mathbb{R}^m \}$ ———— // ————

Ex: $W = \{$ ... $f: \mathbb{R}^n \to \mathbb{R}^m \}$

Ex: $V = \{ \text{all } f: \mathbb{R}^n \to \mathbb{R}^m \}$ ———— // ————

$\hookrightarrow f, g: \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow (f+g)(x) = f(x) + g(x)$

$\hookrightarrow \lambda \in \mathbb{R}, f: \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow (\lambda f)(x) = \lambda f(x)$

$0_V + f = f + 0_V = f$ let $0_V(x) = 0$ i.e. the zero fct

$f + (-f) = 0_V$ let $(-f)(x) = -f(x)$

Non-example: $\{$invertible linear functions $f: \mathbb{R}^2 \to \mathbb{R}^2\}$ NOT a v.s. using usual addition

- does not have a zero-element (b/c 0_V is not invertible)
- sum of invertible functions need not be invertible

$$Id_{\mathbb{R}^2} + (-Id)_{\mathbb{R}^2} = (\text{zero function}) \text{ not invertible}$$

Properties of any vector space V:

- $-v = (-1)v$

opposite element

scalar multiplications

we will prove that RHS satisfies the opposite element axiom

$$v + (-v) = 0_V$$

$$\Downarrow$$

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0_V$$

- $0v = 0_V$

$$0 = 0 + 0$$

$$\Downarrow$$

$0v = (0 + 0)v = 0v + 0v$

$0v = (0+0)v = 0v + 0v$

$\Downarrow$

let w be the opposite of $0v$, add w to the equality above

$$0_V = w + 0v = w + (0v + 0v) = (w + 0v) + 0v = 0_V + 0v = 0v$$

- $\lambda 0_V = 0_V$
- opposite vectors are unique $\left(\forall v, \text{ if } \exists\, v', v'' \text{ such that } \begin{matrix} v+v'=0 \\ v+v''=0 \end{matrix} \right\} \Rightarrow v' = v'' \Big)$

prove these yourselves

DEF 12.2: consider a vector space V

A nonempty subset $W \subseteq V$ is called a **subspace** if

it is closed under $\begin{cases} \text{addition} : \forall v, v' \in W,\ v + v' \in W \\ \text{scalar multiplication} : \forall v \in W, \forall \lambda \in \mathbb{R},\ \lambda v \in W \end{cases}$

(note that such a W must contain 0_V because:

pick $\begin{matrix} v \in W \\ 0 \in \mathbb{R} \end{matrix}$, then $0v = 0_V \in W$)

Ex: $W = V$

$W = \{0_V\}$ (b/c $0_V + 0_V = 0_V$ and $\lambda \cdot 0_V = 0_V$)

$W = \{\text{li}\ldots\ f : \mathbb{R}^n \to \mathbb{R}^m\} \subset V = \{\text{all } f : \mathbb{R}^n \to \mathbb{R}^m\}$

$W = \{\text{linear } f: \mathbb{R} \to \mathbb{R}\} \subseteq V = \{\text{all } f: \mathbb{R} \to \mathbb{R}\}$

THM 12.3: if W is a subspace of a vector space V then W is a vector space in its own right, w.r.t. addition and scalar multiplication inherited from V

- $w, w' \in W \subseteq V \rightsquigarrow w + w' \in W \subseteq V$
- $\lambda \in \mathbb{R},\ w \in W \subseteq V \rightsquigarrow \lambda w \in W \subseteq V$

New topic:

subspaces $W \subseteq \mathbb{R}^n$

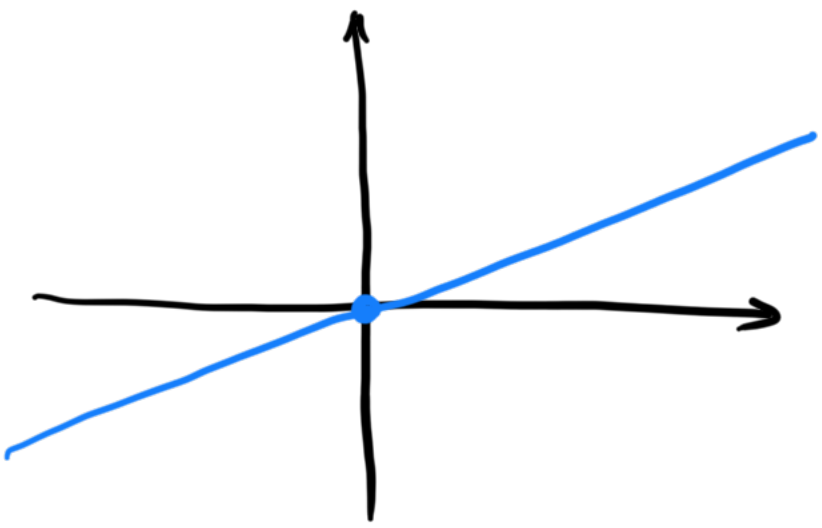
- $\hookrightarrow$ contains $0_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$
- $\hookrightarrow$ is closed under addition
- $\hookrightarrow$ is closed under scalar multiplication

} closed under linear combi

$\forall\, w, w' \in W$
$\forall\, c, c' \in \mathbb{R}$, $cw + c'w' \in W$

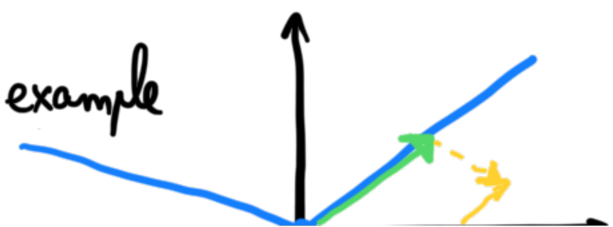
Ex: subspaces of $\mathbb{R}^2$: $\mathbb{R}^2$, $\{0\}$

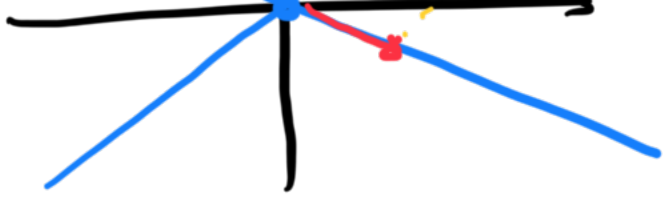
lines that pass through origin



Non-example:

Non-example





Ex : subspaces of $\mathbb{R}^3$: $\mathbb{R}^3$, $\{0\}$

lines which pass through 0

planes which pass through 0

DEF 12.4 : in any vector space V,

take $v_1, \dots, v_k \in V$ for $k \geq 1$

$$\mathrm{Span}\{v_1, \dots, v_k\} = \{ c_1 v_1 + \dots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R} \} \subseteq V$$

Prop: $\mathrm{span}\{v_1, \dots, v_k\}$ is a subspace of V, $\forall v_1, \dots, v_k$

to prove this, we must show that

- span is closed under addition

$$(c_1 v_1 + \dots + c_k v_k) + (c_1' v_1 + \dots + c_k' v_k) = (c_1 + c_1') v_1 + \dots + (c_k + c_k') v_k$$

- span is closed under scalar multiplication

$$\lambda (c_1 v_1 + \dots + c_k v_k) = (\lambda c_1) v_1 + \dots + (\lambda c_k) v_k$$